Stability of the Cournot equilibrium for a Cournot oligopoly model with n competitors

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Abstract

In this paper a Cournot-like model is constructed with an iso-elastic demand function for n competitors. The Cournot equilibrium is constructed for general constant unit costs. Finally, it is proved that for identical unit costs the Cournot point is a sink for two or three competitors and a saddle for more than four players.

Keywords: Cournot equilibrium, dynamical system, sink, source, saddle
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1. Introduction

There are two opposing market forms in economics: competition and monopoly. In the case of competition, firms are numerous and hence small in comparison to the size of the total market, and they consider the market price to be exogenously determined. In a monopoly, only one firm supplies the market and supply influences the market price appreciably.

An *oligopoly* is a market form in which a market has a dominant influence on a small number of sellers (oligopolists). Since there are few sellers, each oligopolist is likely to be aware of the actions of the others. Each seller has an influence on, and is influenced by, the decisions of the other oligopolists. Hence, the planning of each oligopolist needs to take into account the responses of the other competitors. In an oligopoly, there are at least two firms controlling the market. If there are two sellers, it is called a *duopoly*;

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while if there are three competitors, it is known as a *triopoly.* The first treatment of oligopoly was proposed by A. Cournot, in 1838 [4], for a duopoly. Significant additions to the theory were made exactly one hundred years later by H. von Stackelberg [12].

A question in microeconomics is whether an increase in the number of competitors in a market defines a path to perfect competition. It was stated by [13] (see also [8] page 237) that the oligopoly model produced under constant marginal costs with a linear demand function is neutrally stable for three competitors and unstable for more than three competitors. The argument for this fact can be found in [11]. As discussed in [11], linear demand functions are very easy to use, but they do not avoid negative supplies and prices, so it is possible to use them only for the study of local behavior. This problem can be solved by using nonlinear demand functions such as piecewise linear functions or other more complex functions, one of which was suggested by [9] for a dupoly and later by [10] for a triopoly using iso-elastic demand functions. These types of demand function were later studied by [1] and [2] for a nonlinear (iso-elastic) demand function and constant marginal costs and it was concluded that this Cournot model for n competitors is neutrally stable if $n = 4$ and is unstable if the number of competitors is greater than five (see also [11]).

The main aim of this paper is to consider Cournot points and discuss their stability while the number of players is increasing for the model with an iso-elastic demand function and under the assumption that the firms' costs are identical. The terminology of dynamical systems is used, that is the Cournot point is identified as a fixed one.

A discrete dynamical system is an ordered pair (X, f) where X is standardly taken to be a compact metric space and f is a continuous map from X to, but not necessarily onto, X. So, X is *invariant* under f, that is $f(X) \subset X$. A point $x \in X$ is *fixed* if $f(x) = x$ and a set of all fixed points of the map f is denoted by $\mathrm{Fix}(f)$. For $t \in \mathbb{N}$, the *t-th iterate* of f is the t-fold composition $f^t = f \circ \ldots \circ f$, where f^0 is the identity map.

The paper is organized as follows. In the second section the Cournot iso-elastic model with n competitors is derived, and the model is constructed as a discrete dynamical system $(\mathbb{X}_n, F_n)$. In the third section it is proved that the Cournot point is a sink for $n = 2, 3$ and is a saddle for $n > 4$. It is known that for $n = 4$ the Cournot point is neutrally stable (see e.g. [1]

or [11]). Finally, in the last section the analogous results are discussed that were given in the previous section on a model with the assumption that firms compete with their closest competitors in either direction.

2. The Cournot iso-elastic model for N competitors

The following construction is inspired by the work of T. Puu [9] who constructed the model for two competitors (later in [10] for three players). This model could be extended for n firms.

Assuming that the level of demand is reciprocal to price p, this represents an "iso-elastic" demand function reflecting a case where consumers always spend a constant sum on the commodity, regardless of price. Inverting the demand function gives

$$p = \frac{1}{x_1 + x_2 + \cdots + x_n}, \tag{1}$$

where the total quantity in the denominator is the sum of the supplies and x_i are competitors, for $i \in \{1, 2, \ldots, n\}$.

The revenues of these firms equal price times quantity

$$px_i = \frac{x_i}{x_1 + x_2 + \cdots + x_n}. \tag{2}$$

Assuming that the firms operate under constant unit costs c_i, so $c_i > 0$ for any i. Their total costs are $c_i x_i$. So, the profits become

$$\Pi_i(x_1, x_2, \ldots, x_n) = \frac{x_i}{x_1 + x_2 + \cdots + x_n} - c_i x_i. \tag{3}$$

In order to maximize profits for the firm put a partial derivative of (3) with respect to x_i equal to zero

$$\frac{\partial \Pi_i}{\partial x_i} = 0, \tag{4}$$

and these *reaction functions* are derived

$$\begin{aligned}
x_1(x_2, x_3, \ldots, x_n) &= \sqrt{\frac{x_2 + x_3 + \ldots + x_n}{c_1}} - (x_2 + x_3 + \ldots + x_n), \\
x_2(x_1, x_3, \ldots, x_n) &= \sqrt{\frac{x_1 + x_3 + \ldots + x_n}{c_2}} - (x_1 + x_3 + \ldots + x_n), \\
&\vdots \\
x_n(x_2, x_3, \ldots, x_{n-1}) &= \sqrt{\frac{x_2 + x_3 + \ldots + x_{n-1}}{c_n}} - (x_2 + x_3 + \ldots + x_{n-1}).
\end{aligned} \tag{5}$$

Introducing the adjustment process explicitly, from (5) the following are obtained

$$\begin{aligned} x_1^t &= \sqrt{\frac{x_2^{t-1}+x_3^{t-1}+\ldots+x_n^{t-1}}{c_1}} - (x_2^{t-1}+x_3^{t-1}+\ldots+x_n^{t-1}), \\ x_2^t &= \sqrt{\frac{x_1^{t-1}+x_3^{t-1}+\ldots+x_n^{t-1}}{c_2}} - (x_1^{t-1}+x_3^{t-1}+\ldots+x_n^{t-1}), \\ &\vdots \\ x_n^t &= \sqrt{\frac{x_2^{t-1}+x_3^{t-1}+\ldots+x_{n-1}^{t-1}}{c_n}} - (x_2^{t-1}+x_3^{t-1}+\ldots+x_{n-1}^{t-1}). \end{aligned} \tag{6}$$

Thus, a dynamical system is deduced

$$(\mathbb{X}_n, F_n) \tag{7}$$

defined as follows for $n \geq 2$.

Denote

$$\mathbb{x} = (x_1, x_2, x_3, \ldots, x_n)$$

and define operator $({}^i\bullet)$

$$({}^i\mathbb{x}) = \sum_{j \in N_i} x_j$$

where $N = \{1, 2, 3, \ldots, n\}$ and $N_i = N \setminus \{i\}$.

Firstly, set

$$F_n : \mathbb{R}^n \to \mathbb{R}^n$$

defined by

$$\mathbb{x} \mapsto \left(\sqrt{\frac{({}^1\mathbb{x})}{c_1}} - ({}^1\mathbb{x}), \sqrt{\frac{({}^2\mathbb{x})}{c_2}} - ({}^2\mathbb{x}), \ldots, \sqrt{\frac{({}^n\mathbb{x})}{c_n}} - ({}^n\mathbb{x}) \right).$$

Secondly, the invariant set $\mathbb{X}_n$ is found. It is clear that the domain of F_n is

$$D_n = \left\{ ({}^i\mathbb{x}) \geq 0, \text{ for any } i \right\}.$$

Such a model is being considered so that the process can be repeated, thus focussing on a subset of D_n for which $F_n^t(\mathbb{x}) \in D_n$ for any $t \geq 0$. Such points in D_n are *admissible* (see [3]). Unfortunately, not all admissible points are

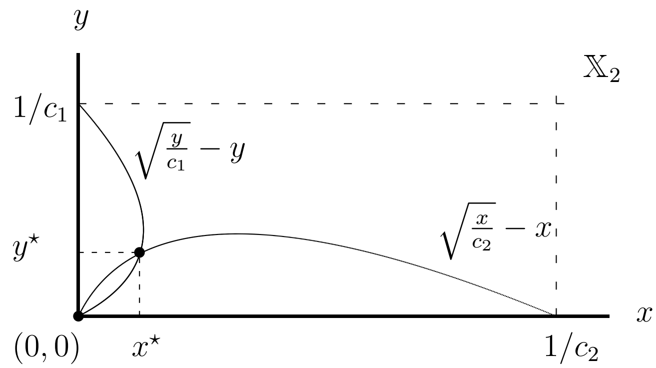

Figure 1: Cournot equilibrium in the second dimension

meaningful. Since economic interpretation of a point with negative value is not acceptable the attention has to be restricted to *feasible* points. The first iteration needs to be taken into account. To do this, this system of equations has to be solved

$$({}^i\mathbb{x}) = \frac{1}{c_i}, \tag{8}$$

Solving system (8) gives the solution

$$x_i = \frac{1}{n-1}\left({}^i(1/\mathbb{c})\right) + (2-n)c_i, \tag{9}$$

where $1/\mathbb{c} = (1/c_1, 1/c_2, \ldots, 1/c_n)$. Denoting x_i by a_i, a set is obtained, such that all their points are feasible for the first iteration. Repeating this process for all iterations gives

$$\mathbb{X}_n \subset [0, a_1] \times [0, a_2] \times \ldots \times [0, a_n] \subset D_n \tag{10}$$

which is invariant under F_n, see Figure 1 for the second-dimensional case. Unfortunately, it is not easy to ascertain the set $\mathbb{X}_n$ explicitly.

Notice that

$$F_n(a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0),$$

and that the spaces $\mathbb{X}_3$, $\mathbb{X}_4$ are not well constructed e.g. in [6] and [7]. The space $\mathbb{X}_2 = [0, 1/c_2] \times [0, 1/c_1]$ is implicitly given in [14], and this can be compared with (9) and (10). It is also easy to see that a fixed point $\mathbb{x}^\star$ of F_n is the *Cournot equilibrium* (Nash equilibrium of the game) (see e.g. [3]).

3. Stability of the Cournot equilibrium of the model

A fixed point p for $f : \mathbb{R}^n \to \mathbb{R}^n$ is called *hyperbolic* if $Df(p)$ has no eigenvalues on the unite circle, where $Df(p)$ is the Jacobian matrix of f at the point p. Such a hyperbolic point p is

1. a *sink* fixed point if all eigenvalues of $Df(p)$ are less than one in absolute value,
2. a *source* fixed point if all eigenvalues of $Df(p)$ are greater than one in absolute value,
3. a *saddle* fixed point otherwise, i.e., if some eigenvalues of $Df(p)$ are less and some larger than one in absolute value.

Proposition 1 ([5]). *Supposing that $f : \mathbb{R}^n \to \mathbb{R}^n$ has a sink fixed point p. Then there is an open set containing p in which all points tend to p under forward iteration of f.*

The largest such open set in $\mathbb{R}^n$ is called the *stable set* of p and is denoted by $W^s(p)$.

Proposition 2 ([5]). *Supposing that $f : \mathbb{R}^n \to \mathbb{R}^n$ has a source fixed point p. Then there is an open set containing p in which all points tend to p under backward iteration of f.*

The largest such open set in $\mathbb{R}^n$ is called the *unstable set* of p and is denoted by $W^u(p)$.

The stability of the Cournot point has been discussed for two or three players. The case for two competitors was discussed in [14]. For $n = 2$

$$\mathrm{Fix}(F_2) = \left\{ (0,0), \left(\frac{c_2}{(c_1+c_2)^2}, \frac{c_1}{(c_1+c_2)^2} \right) \right\}.$$

In [3] the situation is for $n = 3$

$$\mathrm{Fix}(F_3) = \left\{ (0,0,0), \left(\frac{2(-c_1+c_2+c_3)}{(c_1+c_2+c_3)^2}, \frac{2(c_1-c_2+c_3)}{(c_1+c_2+c_3)^2}, \frac{2(c_1+c_2-c_3)}{(c_1+c_2+c_3)^2} \right) \right\}.$$

Note that the first part of Theorem 2 in [7] is not correct. The mistake is in the section "Proof of Theorem 2". Taking $c_3 = c_1 + c_2$ it is concluded "does not provide any fixed point different from the origin". Yet, the second fixed point is $(c_2/(c_1+c_2)^2, c_1/(c_1+c_2)^2, 0)$.

Theorem 3. *Let $n \geq 2$ and $(\mathbb{X}_n, F_n)$ be a dynamical system (7). Then*

$$\mathrm{Fix}(F_n) = \{\mathbb{f}_0, \mathbb{f}_1\}$$

where

$$\mathbb{f}_0 = (0, 0, \ldots, 0),$$

$$\mathbb{f}_1 = \left(\frac{(n-1)({}_1\mathbb{c})}{\mathbb{c}^2}, \frac{(n-1)({}_2\mathbb{c})}{\mathbb{c}^2}, \ldots, \frac{(n-1)({}_n\mathbb{c})}{\mathbb{c}^2} \right),$$

$$\mathbb{c} = \sum_{j \in \{1,2,\ldots,n\}} c_j,$$

$$({}_i\mathbb{c}) = \sum_{j \in N_i} c_j + (2-n)c_i.$$

Proof. The following is going to be solved

$$F_n(\mathbb{x}) = \mathbb{x}. \tag{11}$$

It is easy to see that $\mathbb{f}_0 = (0, 0, \ldots, 0)$ is a trivial solution. Assuming that $\mathbb{x} \neq (0, 0, \ldots, 0)$. From (11) a system of equations is elicited

$$\sqrt{\frac{({}^i\mathbb{x})}{c_i}} - ({}^i\mathbb{x}) = x_i, \tag{12}$$

and hence

$$\sqrt{\frac{({}^i\mathbb{x})}{c_i}} = \mathbb{x}. \tag{13}$$

Giveing

$$\frac{({}^1\mathbb{x})}{c_1} = \frac{({}^2\mathbb{x})}{c_2} = \ldots = \frac{({}^n\mathbb{x})}{c_n} \tag{14}$$

a system of $(n-1)$ linear equations with n parameters. Now, parameters c_i are expressed, and each c_i depends on all x_j where $j \in N_i$. Putting these parameters into (12) the following solution is obtained

$$\mathbb{f}_1 = \left(\frac{(n-1)({}_1\mathbb{c})}{\mathbb{c}^2}, \frac{(n-1)({}_2\mathbb{c})}{\mathbb{c}^2}, \ldots, \frac{(n-1)({}_n\mathbb{c})}{\mathbb{c}^2} \right),$$

$$\mathbb{c} = \sum_{j \in \{1,2,\ldots,n\}} c_j,$$

$$(_i\mathbb{c}) = \sum_{j\in N_i} c_j + (2-n)c_i$$

thus finalising the proof.

It is worth noting how to calculate the image of $\mathbb{f}_1$. Firstly, the following is simplified

$$(n-1)\sum_{j\in N_1}(_j\mathbb{c}) = (n-1)\sum_{k\in N_2}\left(\sum_{j\in N_k} c_j + (2-n)c_k\right) = (n-1)^2 c_1. \quad (15)$$

From (15) the first coordinate of $F_n(\mathbb{f}_1)$ is equal to the first coordinate of $\mathbb{f}_1$:

$$\sqrt{\frac{(n-1)^2 c_1/(_2\mathbb{c})}{c_1}} - \frac{(n-1)^2 c_1}{\mathbb{c}^2} = \frac{(n-1)(_1\mathbb{c})}{\mathbb{c}^2}.$$

The remaining coordinates can be verified analogously.

From now on it is assumed that the oligopolists' products have identical marginal costs, that is $c_i = c$ for any i. Then using Theorem 3 the following is deduced

$$\mathrm{Fix}(F_n) = \left\{(0,0,\ldots,0), \left(\frac{(n-1)}{cn^2}, \frac{(n-1)}{cn^2}, \ldots, \frac{(n-1)}{cn^2}\right)\right\}.$$

The following lemma plays a key role in the proof of the main result, Theorem 5.

Lemma 4. *Let A be an $n \times n$ matrix*

$$A = \begin{pmatrix} a & 1 & 1 & \ldots & 1 \\ 1 & a & 1 & \ldots & 1 \\ 1 & 1 & a & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & a \end{pmatrix}.$$

Then $\det(A) = (a+n-1)(a-1)^{n-1}$.

PROOF. It is easy to see that

$$\begin{array}{ll} \alpha_1 = (1, 1, 1, \ldots, 1, 1) & e_1 = (a + n - 1), \\ \alpha_2 = (1, -1, 0, \ldots, 0, 0) & e_2 = (a - 1), \\ \alpha_3 = (0, 1, -1, \ldots, 0, 0) & e_3 = (a - 1), \\ \quad\vdots & \quad\vdots \\ \alpha_n = (0, 0, 0 \ldots, 1, -1) & e_n = (a - 1) \end{array}$$

are eigenvectors with corresponding eigenvalues of the matrix A, thus completing the proof.

Theorem 5. *If $c_i = c$ for any i, then for a dynamical system $(\mathbb{X}_n, F_n)$ (7):*

(i) $\dim W^u(\mathbb{f}_0) = n$ *and* $\dim W^s(\mathbb{f}_0) = 0$ *for any* $n \geq 2$,
(ii) $\dim W^u(\mathbb{f}_1) = 0$ *and* $\dim W^s(\mathbb{f}_1) = n$ *for* $n = 2, 3$,
(iii) for $n = 4$ *the point* $\mathbb{f}_1$ *is not hyperbolic, moreover* $\dim W^s(\mathbb{f}_1) \geq 3$,
(iv) $\dim W^u(\mathbb{f}_1) = 1$ *and* $\dim W^s(\mathbb{f}_1) = n - 1$ *for any* $n > 4$.

PROOF. The Jacobi matrix of F_n at the point $\mathbb{x}$ is

$$DF_n(\mathbb{x}) = \begin{pmatrix} 0 & \frac{1}{2\sqrt{c}}\left({}^1\mathbb{x}\right)^{-1/2} - 1 & \ldots & \frac{1}{2\sqrt{c}}\left({}^1\mathbb{x}\right)^{-1/2} - 1 \\ \frac{1}{2\sqrt{c}}\left({}^2\mathbb{x}\right)^{-1/2} - 1 & 0 & \ldots & \frac{1}{2\sqrt{c}}\left({}^2\mathbb{x}\right)^{-1/2} - 1 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2\sqrt{c}}\left({}^n\mathbb{x}\right)^{-1/2} - 1 & \frac{1}{2\sqrt{c}}\left({}^n\mathbb{x}\right)^{-1/2} - 1 & \ldots & 0 \end{pmatrix}.$$

Firstly, partial derivatives of F_n at the point $\mathbb{f}_0$ that appear in the Jacobi matrix tend to infinity while $\mathbb{x}$ tends to $\mathbb{f}_0$. So, the point $\mathbb{f}_0$ is unstable in all directions, thus proving (i).

Secondly, constructing a Jacobi matrix of F_n at the point $\mathbb{f}_1$ and looking for its eigenvalues, the following is obtained

$$DF_n(\mathbb{f}_1) = \begin{pmatrix} 0 & \frac{n}{2(n-1)} - 1 & \ldots & \frac{n}{2(n-1)} - 1 \\ \frac{n}{2(n-1)} - 1 & 0 & \ldots & \frac{n}{2(n-1)} - 1 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n}{2(n-1)} - 1 & \frac{n}{2(n-1)} - 1 & \ldots & 0 \end{pmatrix}.$$

Now,

$$\det\left(DF_n(\mathbb{f}_0)-\lambda I\right)=\left(\frac{2-n}{2(n-1)}\right)^n\det(M)$$

where

$$M=\begin{pmatrix} -\lambda\frac{2(n-1)}{2-n} & 1 & 1 & \dots & 1\\ 1 & -\lambda\frac{2(n-1)}{2-n} & 1 & \dots & 1\\ 1 & 1 & -\lambda\frac{2(n-1)}{2-n} & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \dots & -\lambda\frac{2(n-1)}{2-n}\end{pmatrix}.$$

So, according to Lemma 4 the following may be now deduced

$$\det\left(DF_n(\mathbb{f}_1)-\lambda I\right)=\left(\frac{2-n}{2(n-1)}\right)^n\left(-\lambda\frac{2(n-1)}{2-n}+n-1\right)\left(-\lambda\frac{2(n-1)}{2-n}-1\right)^{n-1}.$$

There are two different real roots

$$\lambda_1=1-\frac{n}{2},$$

$$\lambda_2=\frac{n-2}{2(n-1)}$$

where the first one is with one multiple and the second one with $n-1$ multiple.

Their absolute values are

	$n=2,3$	$n=4$	$n>4$
$\lvert\lambda_1\rvert$	<1	$=1$	>1
$\lvert\lambda_2\rvert$	<1	<1	<1

To finalize the proof it is sufficient to use vectors in the proof Lemma 4 to generate suitable subspaces of the range space $\mathbb{R}^n$, proving (ii), (iii) and (iv).

Corollary 6. *The fixed point $\mathbb{f}_0$ of F_n is a source for any $n\geq 2$ and $c_i=c$.*

Corollary 7. *The Cournot point $\mathbb{f}_1$ of F_n is a sink for any $n=2,3$ and $c_i=c$.*

Corollary 8. *The Cournot point $\mathbb{f}_1$ of F_n is a saddle for any $n>4$ and $c_i=c$.*

4. Conclusion

The Cournot point was constructed for $(\mathbb{X}_n, F_n)$ and for general unit costs in Theorem 3 and its stability discussed under the additional assumption that the unit costs are identical for all firms in Theorem 5. The Cournot point is a sink for two and three competitors according to Corollary 7 and is a saddle for more than four competitors by Corollary 8, under the assumption of constant marginal costs. It is known that for $n = 4$ the Cournot point is neutrally stable (see e.g. [1] or [11]). Unfortunately, it is not so easy to discuss stability for n competitors in the general case (c_i are different for each i); the situation is more complex and needs in-depth study for each case. The general situation for a triopoly was studied in [3].

It is possible to construct a model "à la Cournot" with the assumption that the firms compete with their closest competitors in either direction. Such systems are equal to $(\mathbb{X}_n, F_n)$ for $n = 2, 3$. The system for four competitors was introduced in [6] where (x_1, x_2, x_3, x_4) is mapped to (y_1, y_2, y_3, y_4) and

$$y_1 = \sqrt{\frac{x_2 + x_4}{c_1}} - (x_2 + x_4), \ y_2 = \sqrt{\frac{x_1 + x_3}{c_2}} - (x_1 + x_3),$$

$$y_3 = \sqrt{\frac{x_2 + x_4}{c_3}} - (x_2 + x_4), \ y_4 = \sqrt{\frac{x_1 + x_3}{c_4}} - (x_1 + x_3).$$

It is also possible to extend this idea to dimension n and get Cournot equilibria under the assumption $c_i = c$ for any i. In this case

$$\mathrm{Fix}(F_n) = \left\{(0, 0, \ldots, 0), \left(\frac{2}{9c}, \frac{2}{9c}, \ldots, \frac{2}{9c}\right)\right\}.$$

Unfortunately, it is not possible to use ideas of (12), (13) and (14) from the proof of Theorem 3 to get the Cournot point for general c_i because the model is more complex. So, how to solve this problem for general number of competitors, is still an unsolved question.

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